Spatial Statistics

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SCSS, May 5th 2021
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Why Spatial Statistics?

Some meteorological data: clear spatial dependence.

A random field can be simply understood as a family of random variables $Z(x)$ defined over an indexing space $X$.

Applications

- Medical imagine
- Computer graphics
- Meteorology, Climatology, Environmental science.
Cosmic Microwave Background (CMB) radiation

- Nobel Prizes for Physics in 1978 and in 2006.
- The main interest in Cosmology.

- Statistical Challenges in Modern Astronomy; Book series starting in 1991 in Penn State.
• Consequence of the mechanism of Big Bang.
• The Universe is embedded in a uniform radiation, that provides pictures of its state nearly $1.37 \times 10^{10}$ years ago!
• Exactly CMB radiation: the oldest electromagnetic radiation in the Universe.
Cosmic Microwave Background (CMB) radiation


- Issue for data analysis: Full-Sky maps not fully reliable (masked parts of the sky).
We interpret CMB radiation as a realization of an isotropic RF of finite variance.

“Einstein cosmological principle” $\Rightarrow$ Isotropy.

Loosely, on sufficiently large distance scales the Universe looks identical everywhere in the space (homogeneity) and appears the same in every direction (isotropy).

The prevailing models for early BB dynamics, predict the random fluctuations to be Gaussian, or quadratic/cubic powers of a GRF.
Formally

Definition

Let \((\Omega, \mathcal{F}, \mathbb{P})\) a probability space and \(X\) a topological measure space.
A Random field \(\{Z(x, \omega) : x \in X, \omega \in \Omega\}\) is a function \(Z : X \times \Omega \to \mathbb{R}\), which is \((\text{Borel}(X) \otimes \mathcal{F})\)-measurable.

In Spatial Statistics the index set \(X\) represents some space domain \(X = \mathbb{R}^d, X = S^d, X = M\).
Challenge

Random Fields, give answers to problems rising in a wide range of areas in science and technology!

Challenge: A rigorous study of Random Fields on manifolds.
Random fields on $\mathbb{S}^2$

Consider a random field $Z(x), \, x \in \mathbb{S}^2$.

Assumptions:
- Isotropic.
- Zero mean.

**Karhunen-Loéve expansion:** $Z(x)$ can be represented as

$$Z(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \quad x \in \mathbb{S}^2, \quad (1)$$

$Y_{\ell m}$: spherical harmonics — an orthonormal system for $L^2(\mathbb{S}^2)$ — and

$$a_{\ell m} = \int_{\mathbb{S}^2} Z(x) \bar{Y}_{\ell m}(x) dx. \quad (2)$$
Covariance function

On such a \( \{ Z(\mathbf{x}) : \mathbf{x} \in S^2 \} \):

\[
\text{Cov}(Z(x), Z(y)) = \mathbb{E}(Z(x)Z(y)) = K(\rho(x, y)),
\]

where

\[
K(\theta) = \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell + 1}{2\pi} P_\ell(\cos \theta),
\]

where \( P_\ell \): Legendre polynomials and

\[
A_\ell := \mathbb{E}(|a_{\ell m}|^2) \quad \text{Angular power spectrum.}
\]

To ensure finite variance

\[
\sigma^2 := \sum_{\ell=0}^{\infty} A_\ell (2\ell + 1) < \infty.
\]
Leading contributions


Transfer the study of the random field, to its covariance function and from this, to the angular power spectrum!

Directions:

- Approximation
- Regularity
- Continuity
- SPDEs
- Simulations
- Applied Spatial Statistics: Cosmology and Environmental science.
Random fields on the sphere


How do we expand the developments?

- Relaxing-modifying assumptions.
- Isotropy?
- Target manifold?
- Adding variables into the study.
- Spatiotemporal Statistics.
Why do we generalize?

- Are our assumptions proper?
- Did we include in the study everything we need?
- Phenomena lead to new setups.
Why do we generalize?
Definition

A metric space \((M, \rho)\) is called two-point homogeneous when:

For every \((x_1, x_2) \in M \times M\) and \((y_1, y_2) \in M \times M\), with

\[ \rho(x_1, x_2) = \rho(y_1, y_2), \]

there exists an isometry mapping \(x_i\) to \(y_i\), \(i = 1, 2\).

Spatial Statistics
Compact two-point homogeneous spaces

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Where the Laplace-Beltrami operator attains the eigenvalues

$$\lambda_\ell := \ell(\ell + \alpha + \beta + 1), \quad \ell \geq 0,$$  

the basis of the $\lambda_\ell$-eigenspace: $\{Y_{\ell,m}, \ 1 \leq m \leq h(\mathcal{M}^d, \ell)\}$,

$$h(\mathcal{M}^d, \ell) := \frac{(2\ell + \alpha + \beta + 1)\Gamma(\beta + 1)\Gamma(\ell + \alpha + \beta + 1)\Gamma(\ell + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + 2)\ell!\Gamma(\ell + \beta + 1)}.$$
Isotropic random fields


\[ \left\{ Z(x) : x \in \mathcal{M}^d \right\} \text{ on } (\Omega, \mathcal{F}, \mathbb{P}) \]

- Real valued.
- Zero mean and finite variance.
- Gaussian.
- Isotropic.
Karhunen-Loéve expansion

\[ Z(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{\infty} \sqrt{\frac{\nu_{\ell}}{h(M^d, \ell)}} X_{\ell,m} Y_{\ell,m}(x) \]  

with convergence in \( L^2(\Omega, L^2(M^d)) \).

- \( X_{\ell,m} \) is a sequence of centered uni-variate independent random variables.
- The (power) spectrum coefficients \( \nu_{\ell} \) satisfy

\[ \nu_{\ell} \geq 0 \quad \text{and} \quad \sum_{\ell=0}^{\infty} \nu_{\ell} < \infty. \]
**Covariance**

\[
K_Z(x, y) = \mathbb{E}(Z(x)Z(y)) - \mathbb{E}(Z(x))\mathbb{E}(Z(y)), \quad x, y \in \mathcal{M}^d
\]

\[
= k_Z(\cos(\rho(x, y))),
\]

where \(k_Z : [-1, 1] \rightarrow \mathbb{R}\), satisfies

\[
k_Z(t) = \sum_{\ell=0}^{\infty} \nu_\ell \frac{P_\ell^{(\alpha, \beta)}(t)}{P_\ell^{(\alpha, \beta)}(1)}, \quad t \in [-1, 1],
\]

where \(P_\ell^{(\alpha, \beta)}\) denotes the Jacobi polynomial of order \(\ell\), associated with the pair \((\alpha, \beta)\).
The behavior of the RF is governed by the spectrum!
Norm

- Counting the size of objects on a vector space.
- On $\mathbb{R}^2$: Let $\vec{u} = (u_1, u_2)$, then

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}.$$  \hspace{1cm} (11)

- Metric or distance. Counts how far are the elements of a space, from each other.

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$  \hspace{1cm} (12)
Norms for functions

- We need to count distances \( d(f, g) = \|f - g\| \), between functions.
- Let \( p \geq 1 \) and \( g : \mathcal{M} \to \mathbb{R} \), then \( g \in L^p(\mathcal{M}) \) (Lebesgue) if-f

\[
\|g\|_p := \left( \int_{\mathcal{M}} |g(x)|^p \, dx \right)^{1/p} < \infty.
\] (13)

- When \( p = 1 \) and \( \mathcal{M} = [0, 1] \), then

\[
\|g\|_1 = \int_0^1 |g(x)| \, dx = \text{Area plot x-axis}.
\]
How do we measure the smoothness?

Let \( \alpha, \beta > -1 \). A function \( f : [-1, 1] \rightarrow \mathbb{R} \), belongs to the weighted

- **Lebesgue space** \( L^2_{(\alpha, \beta)} := L^2_{(\alpha, \beta)}[-1, 1] \), when

\[
\|f\|_{L^2_{(\alpha, \beta)}}^2 := \int_{-1}^{1} |f(t)|^2 (1 - t)^\alpha (1 + t)^\beta \, dt < \infty. \tag{14}
\]

Note that \( \{P^{(\alpha, \beta)}_\ell : \ell \geq 0\} \) is an orthogonal basis.

- **Sobolev space** \( W^n = W^n_{(\alpha, \beta)}, \ n \in \mathbb{N} \), when

\[
\|f\|_{W^n}^2 = \|f\|_{W^n_{(\alpha, \beta)}}^2 := \sum_{m=0}^{n} \|f^{(m)}\|_{L^2_{(\alpha+m, \beta+m)}}^2 < \infty. \tag{15}
\]

- Of course \( L^2 \supset W^0 \supset W^1 \supset W^2 \supset \cdots \).
By Kerkyacharian et al (2018), the smoothness of a Random Field is equivalent with the smoothness of the covariance (kernel).

Lang-Schwab (2015), found the Sobolev norms for $S^d$.

Cleanthous et al (2020), express the Sobolev smoothness of the RF in terms of the angular power spectrum.

The summability of the ps, guarantees already the $L^2_{(\alpha, \beta)}$ membership of the ck and therefore the RF.
Weighted $\ell^2$-summability of the spectrum, is equivalent with regularity measured in terms of Sobolev spaces!

**Theorem**

Let $n \in \mathbb{N}$ and \( \{Z(x) : x \in \mathcal{M}^d\} \) an isotropic GRF. Then \( k_Z \in W^n \) if and only if

$$\| k_Z \|_{W^n}^2 \sim \sum_{\ell=0}^{\infty} \nu^2_{\ell} (\ell + 1)^{-2\alpha + 2n - 1} < \infty.$$  \((16)\)

- We proved this result for the more general class of interpolation spaces, measuring non-integer smoothness.
Smooth Random fields

- On $\mathcal{M}^d = S^2$, the above theorem translates as follows:

$$ \| k_Z \|^2_{W^s} \sim \sum_{\ell=0}^{\infty} A_\ell^2 (\ell + 1)^{2s+1} $$  \hspace{1cm} (17)

- Simply taking $A_\ell = (\ell + 1)^{-\tau}$, we have

$$ k_Z \in W^s \iff \tau > s + 1 $$ \hspace{1cm} (18)

- Below we draw some (simulated) RFs for $\tau_1 = 3$ and $\tau_2 = 5$. 
(a) $k_Z \in W^1 \setminus W^2$ and (b) $k_Z \in W^3 \setminus W^4$. 
Hölder spaces

- Let $n \in \mathbb{N}_0$. We denote by $C^n = C^n[-1,1]$ the set of all functions $f : [-1,1] \to \mathbb{R}$ such that their derivatives up to order $n$, exist and are continuous.

$$\|f\|_{C^n} := \sum_{m=0}^{n} \sup_{-1 \leq t \leq 1} |f^{(m)}(t)| < \infty. \quad (19)$$

- Let now $N > 0$ be a non integer and let $n := [N]$ be the integer part of $N$. The Hölder space of order $N$, denoted $C^N = C^N[-1,1]$, is defined as the class of all functions $f \in C^n$ such that

$$\|f\|_{C^N} := \|f\|_{C^n} + \sup_{-1 \leq t \neq s \leq 1} \frac{|f^{(n)}(t) - f^{(n)}(s)|}{|t - s|^{N-n}} < \infty. \quad (20)$$
Sample Hölder continuous random fields

Let $\gamma \in (0, 1)$. A random field $Z : \mathcal{M}^d \times \Omega \to \mathbb{R}$ is called:

**sample $\gamma$-Hölder continuous**, when for every $\omega \in \Omega$, the sample function $Z(\cdot, \omega) : \mathcal{M}^d \to \mathbb{R}$ is $\gamma$-Hölder continuous, i.e. there exist a constant $c > 0$ such that

$$|Z(x, \omega) - Z(y, \omega)| \leq c \rho(x, y)\gamma, \quad \text{for every } x, y \in \mathcal{M}^d. \quad (21)$$

**locally sample $\gamma$-Hölder continuous**, when for every $\omega \in \Omega$, and every $z \in \mathcal{M}^d$, there exists a neighbor $V \ni z$ such that the sample function $Z(\cdot, \omega) : V \to \mathbb{R}$ is $\gamma$-Hölder continuous, i.e. there exist a constant $c = c_V > 0$ such that

$$|Z(x, \omega) - Z(y, \omega)| \leq c \rho(x, y)\gamma, \quad \text{for every } x, y \in V. \quad (22)$$
Sample Hölder continuity of the RF

Weighted $\ell^1$-summability of the spectrum, implies Hölder continuity, bounds of the moments of $Z(x) - Z(y)$ and the existence of a Hölder continuous modification!

**Theorem**

Let $N > 0$. Let $\{Z(x) : x \in \mathcal{M}^d\}$ be an isotropic GRF, whose spectrum $(\nu_\ell)_{\ell \in \mathbb{N}_0}$ from (8) satisfies

$$\sum_{\ell=0}^{\infty} \nu_\ell (\ell + 1)^{2N} < \infty. \quad (23)$$

Then, the isotropic covariance kernel $k_Z$ is $N$-Hölder continuous.
Moments of $Z(x) - Z(y)$

**Theorem**

Let $\{Z(x) : x \in \mathcal{M}^d\}$ be an isotropic GRF, whose spectrum $(\nu_\ell)_{\ell \in \mathbb{N}_0}$ satisfies (23) for some $N \in (0, 1]$. Then, for every $p \in \mathbb{N}$, there exists a constant $c = c_{N,p} > 0$ such that for every $x, y \in \mathcal{M}^d$,

$$\mathbb{E}(|Z(x) - Z(y)|^{2p}) \leq c \rho(x, y)^{2pN}.$$  

(24)
A Kolmogorov-Chentsov type theorem

Theorem

Let \( \{Z(x) : x \in \mathcal{M}^d \} \) be an isotropic GRF, whose spectrum \((\nu_\ell)_{\ell \in \mathbb{N}_0}\) satisfies (23) for some \( N \in (0, 1] \). Then, there exists a continuous modification of \( Z \) which is sample Hölder continuous of order \( \gamma \in (0, N) \).
How do we really work with an IRF?

\[ Z(x) := \sum_{\ell=0}^{\infty} \sum_{m=1} h(M^d,\ell) \sqrt{\frac{\nu_\ell}{h(M^d,\ell)}} Y_{\ell,m}(x) X_{\ell,m}, \]

Our PC?

Truncation.
For \( r \in \mathbb{N} \), we set

\[
Z^r(x) := \sum_{\ell=0}^{r} \sum_{m=1}^{\nu_{\ell}} \sqrt{\frac{\nu_{\ell}}{h(M^d, \ell)}} Y_{\ell,m}(x) X_{\ell,m},
\]

which is apparently a truncated version of the expansion (8) of the GRF, \( Z \).

- We count the error \( Z - Z^r \) in the \( \mathbb{P} \)-a.s and in mixed Lebesgue norms.
- The decay on the spectrum, guarantees fast approximation!
Recall that $Z : \Omega \times \mathcal{M}^d \to \mathbb{R}$. We need to measure the integrability in the spatial domain $\mathcal{M}^d$ and the stochastic domain $\Omega$. Let $p, q > 0$. We define the (quasi-)norm

\[ \|Z\|_{p,q} := \|Z\|_{L^p(\Omega; L^q(\mathcal{M}^d))} \]

\[ = \left( \mathbb{E}\|Z(\cdot, \omega)\|_{L^q(\mathcal{M}^d)}^p \right)^{1/p} \]

\[ = \left( \mathbb{E} \left( \int_{\mathcal{M}^d} |Z(x, \omega)|^q dx \right)^{p/q} \right)^{1/p} \]

\[ = \left( \int_{\Omega} \left( \int_{\mathcal{M}^d} |Z(x, \omega)|^q dx \right)^{p/q} d\mathbb{P}(\omega) \right)^{1/p}. \]

E.g.

\[ \|Z\|_{2,2}^2 = \mathbb{E} \int_{\mathcal{M}^d} |Z(x, \omega)|^2 dx. \]
Theorem

Let \( \{Z(x) : x \in \mathcal{M}^d\} \) be an isotropic GRF, whose spectrum decays algebraically with order 1 + \( \varepsilon \), \( \varepsilon > 0 \); i.e., there exist \( c_* > 0 \) and \( \ell_0 \in \mathbb{N} \) such that for all \( \ell \geq \ell_0 \)

\[
\nu_\ell \leq c_* \ell^{-1-\varepsilon}.
\]  

(30)

Then, the series of the truncated RFs \( (Z^r)_r \) converges to the RF \( Z \) in \( L^p(\Omega, L^2(\mathcal{M}^d)) \) for every \( p > 0 \). Moreover there exists a constant \( c = c_{p,\varepsilon} > 0 \) such that

\[
\|Z - Z^r\|_{L^p(\Omega, L^2(\mathcal{M}^d))} \leq cr^{-\varepsilon/2}.
\]

(31)

1. \( \mathbb{P} \)-almost surely and for every \( 0 < \gamma < \varepsilon/2 \), the truncated error is asymptotically bounded by

\[
\|Z - Z^r\|_{L^2(\mathcal{M}^d)} \leq r^{-\gamma}, \quad \mathbb{P}\text{-a.s.}
\]

(32)
Multi-variate random fields on the sphere

Let $S^d := \{ x \in \mathbb{R}^{d+1} : \|x\| = 1 \}$ and $k \in \mathbb{N}$.
A $k$-variate random field $Z : S^d \times \Omega \to \mathbb{R}^k$ is called isotropic when it is of constant mean vector and

$$\text{Cov}(Z(x), Z(y)) = C(\rho(x, y)).$$  \hspace{1cm} (33)

Then

$$C(\theta) = \sum_{n=0}^{\infty} A_n C_n \left( \frac{d-1}{2} \right) (\cos \theta),$$  \hspace{1cm} (34)

where $A_n$: positive definite $k \times k$ matrices and

$$\sum_{n=0}^{\infty} A_n C_n \left( \frac{d-1}{2} \right) (1) \in \mathbb{R}^{k \times k}. $$  \hspace{1cm} (35)
Let \( \{V_n\} \) sequence of independent random vectors, with \( \mathbb{E}(V_n) = 0 \) and diagonal \( \text{Cov}(V_n) \). Let \( U: (d + 1) \)-dimensional random vector uniformly distributed on \( S^d \), independent of \( \{V_n\} \) and \( \{A_n\} \) as in (35).

Then the random field

\[
Z(x) := \sum_{n=0}^{\infty} A_n^{1/2} V_n C_n^{(d-1)/2} (x' U),
\]

is \( k \)-variate isotropic random field of zero mean and

\[
C(\theta) = \sum_{n=0}^{\infty} A_n C_n^{(d-1)/2} (\cos \theta),
\]
Measuring matrices


Let $\mathbf{A}, \mathbf{B}$ $n \times m$ matrices. The Frobenius inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle_F := \text{trace}(\mathbf{A}\mathbf{B}').$$  \hspace{1cm} (38)

This gives the natural norm

$$\|\mathbf{A}\|_F^2 = \langle \mathbf{A}, \mathbf{A} \rangle_F = \sum_{i,j} \alpha_{ij}^2.$$  \hspace{1cm} (39)
How do we approximate $Z(x)$?

Truncation:

$$Z^R(x) := \sum_{n=0}^{R} A_n^{1/2} V_n C_n^{(d-1)/2} (x'U), \quad R \in \mathbb{N}. \quad (40)$$

Target: find conditions st $Z^R \rightarrow Z$ and measure the accuracy!

Recall that in the uni-variate case we had the decay of a sequence (of numbers).

$$\|Z\|_{p,2}^p = \|Z\|_{L^p(\Omega, L^2(S^d; \mathbb{R}^k))}^p = \mathbb{E}\left(\|Z(\cdot, \omega)\|_{L^2(S^d; \mathbb{R}^k)}^p\right)$$

$$= \mathbb{E}\left(\int_{S^d} \|Z(x)\|_F^2 dx\right)^{p/2}, \quad p \geq 1. \quad (41)$$
Approximation

**Theorem**

Let $\mathbf{Z}(x)$ a $k$-variate isotropic random field as in (36) such that

$$\text{trace}(\mathbf{A}_n) \leq c_0 n^{-d+1-\varepsilon}, \quad \text{for some } \varepsilon > 0,$$

then $\{\mathbf{Z}^R(x)\}_R$ converges to $\mathbf{Z}(x)$ and

$$\|\mathbf{Z} - \mathbf{Z}^R\|_p \leq C_0 R^{-\varepsilon/2}, \quad \text{for every } p \geq 1.$$
A Bayesian model for the $C$.

- lower triangular, with nonnegative diagonal $B_n$:

$$A_n = B_nB'_n.$$ \hspace{1cm} (44)

Propose a Bayesian model by assigning priors to $B_n$'s;

*The model:* Let $\tilde{B} := \{\tilde{B}_n\}_{n \geq 0}$ independent random matrices of the same type with $B_n$'s:

For every $n \geq 0$:

- iid diagonal elements
- iid off-diagonal elements

$$\mathbb{E}((\tilde{B}_n)_{11})^2 = \mathbb{E}((\tilde{B}_n)_{21})^2 =: d_n; \quad \sum_{n=0}^{\infty} d_n < \infty.$$ \hspace{1cm} (45)
The posterior

**Theorem**

Let \( x_1, \ldots, x_n \in \mathbb{S}^d \) and

\[
    z := (z(x_1)', \ldots, z(x_n)'),
\]

(46)

sampled from the RF.

The posterior \( \mathbb{P}^z \) of \( \tilde{B} \), exists, it is unique and Lipschitz continuous; small data-change, implies small changes in the posterior distribution.

**Remark:** Application of the model to bivariate meteorological data (Atmospheric pressure, DSRF).
Phenomena evolving temporally

A phenomenon may present an additional time dependence; Spatiotemporal random fields.

\[ Z(x, t), \ x \in M, \ t \in T. \tag{47} \]

- A RF \( \{Z(x, t)\} \): space isotropic and time stationary when \( \text{cov}(Z(x_1, t_1), Z(x_2, t_2)) \) depends only on \( \rho(x_1, x_2) \) and \( (t_2 - t_1) \).

Random fields on $\mathcal{M}^d \times \mathbb{R}$:
- Approximation
- Regularity
- Covariance modeling for the study of Ozone concentration.

C. Doherthy.
Spatiotemporal statistics

Spatiotemporal RFs on $\mathcal{M}^d \times \mathbb{R}$.

Let $\{Z(x, t) : x \in \mathcal{M}^d, t \in \mathbb{R}\}$ space isotropic and time stationary. Then

$$\text{cov}(Z(x_1, t_1), Z(x_2, t_2)) = K_{IS}(\cos \rho(x_1, x_2), t_2 - t_1),$$

where the function $K_{IS} : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ is such that

$$K_{IS}(u, t) = \sum_{n=0}^{\infty} B_n(t) P_n^{(\alpha, \beta)}(u),$$

for a sequence $B_n$ of stationary covariance functions;

$$\sum_{n=0}^{\infty} B_n(t) P_n^{(\alpha, \beta)}(1) \in \mathbb{R}.$$
What if the spatiotemporal phenomenon present time-periodicity? Is $\mathbb{R}$ the ideal time domain?
Periodic phenomena

Wrap the time to a circle!

\[ S^2 \times S^1, \quad (51) \]

is the ideal domain for a spatiotemporal periodic phenomenon on the Earth.


Let \( d_1, d_2 \in \mathbb{N} \), we work on

\[ \mathbb{T}^{d_1, d_2} := S^{d_1} \times S^{d_2}, \quad (52) \]

The setting includes the torus as it is isomorphic with \( S^1 \times S^1 \).
Beyond isotropy

Axial symmetry.
What is next?

- Combinations of the above.
- Other settings.
- More general manifolds.
- Isotropy?
Thank you :) 

Thank you very much for your attention!