Spatial Statistics

Athanasios G. Georgiadis

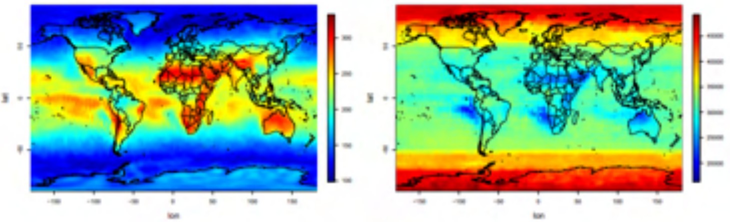
Assistant Professor,
Trinity College of Dublin.

SCSS, May 5th 2021

Contents

Why Spatial Statistics?

Some meteorological data: clear spatial dependence.



- A. Alegria, P. Bisiri, G. Cleanthous, E. Porcu and P. White, Multivariate isotropic random fields on spheres: Nonparametric Bayesian modeling and L^p fast approximations. Elect. J. Stat. 2021, Vol. 15, No. 1, 2360-2392

Random fields

A random field can be simply understood as a family of random variables $Z(x)$ defined over an indexing space X.

Applications

- Medical imagine
- Computer graphics
- Meteorology, Climatology, Environmental science.

Cosmic Microwave Background (CMB) radiation

- Nobel Prizes for Physics in 1978 and in 2006.
- The main interest in Cosmology.
- Applied Statistical Statistics.

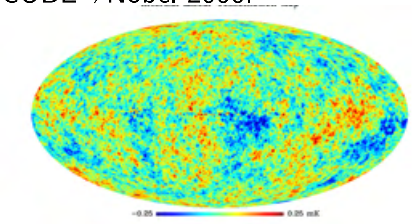
• D. Marinucci, G. Peccati, (2011). Random fields on the sphere: representation, limit theorems and cosmological applications 389. Cambridge University Press.
• Balbi, The music of the Big Bang, (2007).
• Statistical Challenges in Modern Astronomy; Book series starting in 1991 in Penn State.

CMB

- Consequence of the mechanism of Big Bang.
- The Universe is embedded in a uniform radiation, that provides pictures of its state nearly 1.37×10^{10} years ago!
- Exactly CMB radiation: the oldest electromagnetic radiation in the Universe.

Cosmic Microwave Background (CMB) radiation

- Full-Sky maps of radiation (1992) by NASA satellite missions COBE⇒Nobel 2006.



- Issue for data analysis: Full-Sky maps not fully reliable (masked parts of the sky).

Framework

- We interpret CMB radiation as a realization of an isotropic RF of finite variance.
- “Einstein cosmological principle"⇒Isotropy.
- Loosely, on sufficiently large distance scales the Universe looks identical everywhere in the space (homogeneity) and appears the same in every direction (isotropy).
- The prevailing models for early BB dynamics, predict the random fluctuations to be Gaussian, or quadratic/cubic powers of a GRF.

Formally

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and X a topological measure space.
A Random field $\{Z(x, \omega) : \ x \in X, \omega \in \Omega\}$ is a function $Z : X \times \Omega \to \mathbb{R}$, which is $(\text{Borel}(X) \otimes \mathcal{F})$-measurable.

In Spatial Statistics the index set X represents some space domain $X = \mathbb{R}^d$, $X = \mathbb{S}^d$, $X = \mathcal{M}$.

Challenge

Random Fields, give answers to problems rising in a wide range of areas in science and technology!

Challenge: A rigorous study of Random Fields on manifolds.

Random fields on $\mathbb{S}^2$

Consider a random field $Z(x)$, $x \in \mathbb{S}^2$.

Assumptions:

- Isotropic.
- Zero mean.

Karhunen-Loéve expansion: $Z(x)$ can be represented as

$$Z(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \quad x \in \mathbb{S}^2, \tag{1}$$

$Y_{\ell m}$: spherical harmonics —an orthonormal system for $L^2(\mathbb{S}^2)$— and

$$a_{\ell m} = \int_{\mathbb{S}^2} Z(x) \bar{Y}_{\ell m}(x) dx. \tag{2}$$

Covariance function

On such a $\{Z(x) : x \in \mathbb{S}^2\}$:

$$\mathrm{Cov}\big(Z(x), Z(y)\big) = \mathbb{E}\big(Z(x)Z(y)\big) = K\big(\rho(x,y)\big), \tag{3}$$

where

$$K(\theta) = \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{2\pi} P_\ell(\cos\theta), \tag{4}$$

where P_ℓ: Legendre polynomials and

$$A_\ell := \mathbb{E}\big(|a_{\ell m}|^2\big) \quad \text{Angular power spectrum.} \tag{5}$$

To ensure finite variance

$$\sigma^2 := \sum_{\ell=0}^{\infty} A_\ell (2\ell+1) < \infty. \tag{6}$$

Leading contributions

• A. Lang, Ch. Schwab, Isotropic Gaussian random fields on the sphere: regularity, fast simulation and stochastic partial differential equations. Annals Appl. Probability 25 (2015), 30470–3094.

• G. Kerkyacharian, S. Ogawa, P. Petrushev, D. Picard, Regularity of Gaussian Processes on Dirichlet spaces. Constructive Approx. 47, 277–320 (2018).

Transfer the study of the random field, to its covariance function and from this, to the angular power spectrum!

Directions:

- Approximation
- Regularity
- Continuity
- SPDEs
- Simulations
- Applied Spatial Statistics: Cosmology and Environmental science.

Random fields on the sphere

- Lang-Schwab, AoAP (2015).
- Marinucci-Peccati (2011).
- Yadrenko (1983).

- Ultán Doherthy, Isotropic Random Fields on the Sphere, FYT, TCD (2021).

How do we expand the developments?

- Relaxing-modifying assumptions.
- Isotropy?
- Target manifold?
- Adding variables into the study.
- Spatiotemporal Statistics.

Why do we generalize?

- Are our assumptions proper?
- Did we include in the study everything we need?
- Phenomena lead to new setups.

Why do we generalize?

Compact two-point homogeneous spaces

Definition

A metric space $(\mathcal{M}, \rho)$ is called two-point homogeneous when:

For every $(x_1, x_2) \in \mathcal{M} \times \mathcal{M}$ and $(y_1, y_2) \in \mathcal{M} \times \mathcal{M}$, with

$$\rho(x_1, x_2) = \rho(y_1, y_2),$$

there exists an isometry mapping x_i to y_i, $i = 1, 2$.

• Wang, H.-C., Two-point homogenous spaces. Ann. Math. 55, 177–191 (1952).
• Malyarenko, A., Invariant random fields on spaces with a group action. Probability and its Applications. Springer, Heidelberg (2013).

• Cillian Doherthy, Random fields on manifolds, FYT, TCD (2021).

Table

Manifold	$\mathcal{M}^d$	G	K	α	β	Dimension
Unit Sphere	$\mathbb{S}^d$	$\mathrm{SO}(d+1)$	$\mathrm{SO}(d)$	$(d-2)/2$	$(d-2)/2$	$d = 1, 2, \ldots$
Real P.S.	$P^d(\mathbb{R})$	$\mathrm{SO}(d+1)$	$\mathrm{O}(d)$	$(d-2)/2$	$-1/2$	$d = 2, 3, \ldots$
Complex P.S.	$P^d(\mathbb{C})$	$\mathrm{SU}(d+1)$	$S(U(d) \times U(1))$	$(d-2)/2$	0	$d = 4, 6, \ldots$
Quaternionic P.S.	$P^d(\mathbb{H})$	$\mathrm{Sp}(d+1)$	$\mathrm{Sp}(d) \times \mathrm{Sp}(1)$	$(d-2)/2$	1	$d = 8, 12, \ldots$
Cayley P.P.	$P^{16}(\mathrm{Cay})$	$F_{4(-52)}$	$\mathrm{Spin}(9)$	7	3	$d = 16$

Where the Laplace-Beltrami operator attains the eigenvalues

$$\lambda_\ell := \ell(\ell + \alpha + \beta + 1), \quad \ell \geq 0, \tag{7}$$

the basis of the λ_ℓ-eigenspace: $\{Y_{\ell,m},\ 1 \leq m \leq h(\mathcal{M}^d, \ell)\}$,

$$h(\mathcal{M}^d, \ell) := \frac{(2\ell + \alpha + \beta + 1)\Gamma(\beta + 1)\Gamma(\ell + \alpha + \beta + 1)\Gamma(\ell + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + 2)\, \ell!\, \Gamma(\ell + \beta + 1)}.$$

Isotropic random fields

- G. Cleanthous, N, A. Lang and E. Porcu, Regularity, continuity and approximation of isotropic Gaussian random fields on compact two-point homogeneous spaces. Stochastic Processes and their Applications. vol 130 issue 8, August 2020, 4873-4891.

$$\left\{Z(x): \ x \in \mathcal{M}^d\right\} \quad \text{on} \ \ (\Omega, \mathcal{F}, \mathbb{P}).$$

- Real valued.
- Zero mean and finite variance.
- Gaussian.
- Isotropic.

Karhunen-Loéve expansion

$$Z(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\mathcal{M}^d,\ell)} \sqrt{\frac{\nu_\ell}{h(\mathcal{M}^d,\ell)}} X_{\ell,m} Y_{\ell,m}(x) \qquad (8)$$

with convergence in $L^2(\Omega, L^2(\mathcal{M}^d))$.

- $X_{\ell,m}$ is a sequence of centered uni-variate independent random variables.
- The (power) spectrum coefficients ν_ℓ satisfy

$$\nu_\ell \geq 0 \quad \text{and} \quad \sum_{\ell=0}^{\infty} \nu_\ell < \infty. \qquad (9)$$

Covariance

$$\begin{aligned} K_Z(x,y) &= \mathbb{E}(Z(x)Z(y)) - \mathbb{E}(Z(x))\mathbb{E}(Z(y)), \quad x,y \in \mathcal{M}^d \\ &= k_Z\big(\cos(\rho(x,y))\big), \end{aligned}$$

where $k_Z : [-1,1] \to \mathbb{R}$, satisfies

$$k_Z(t) = \sum_{\ell=0}^{\infty} \nu_\ell \frac{P_\ell^{(\alpha,\beta)}(t)}{P_\ell^{(\alpha,\beta)}(1)}, \quad t \in [-1,1], \tag{10}$$

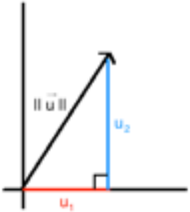
where $P_\ell^{(\alpha,\beta)}$ denotes the Jacobi polynomial of order ℓ, associated with the pair (α,β).

The behavior of the RF is governed by the spectrum!

Norm

- Counting the size of objects on a vector space.
- On $\mathbb{R}^2$: Let $\vec{u} = (u_1, u_2)$, then

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}. \tag{11}$$



- Metric or distance. Counts how far are the elements of a space, from each other.

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|. \tag{12}$$

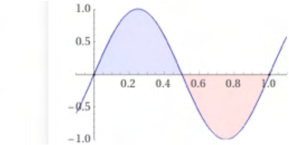
Norms for functions

- We need to count distances $d(f,g) = \|f - g\|$, between functions.
- Let $p \geq 1$ and $g : \mathcal{M} \to \mathbb{R}$, then $g \in L^p(\mathcal{M})$ (Lebesgue) if-f

$$\|g\|_p := \left(\int_{\mathcal{M}} |g(x)|^p dx \right)^{1/p} < \infty. \tag{13}$$

- When $p = 1$ and $\mathcal{M} = [0,1]$, then

$$\|g\|_1 = \int_0^1 |g(x)| dx = \text{Area plot } x\text{-axis.}$$



How do we measure the smoothness?

Let $\alpha, \beta > -1$. A function $f : [-1, 1] \to \mathbb{R}$, belongs to the weighted

- Lebesgue space $L^2_{(\alpha,\beta)} := L^2_{(\alpha,\beta)}[-1, 1]$, when

$$\|f\|^2_{L^2_{(\alpha,\beta)}} := \int_{-1}^{1} |f(t)|^2 (1-t)^\alpha (1+t)^\beta dt < \infty. \quad (14)$$

Note that $\{P_\ell^{(\alpha,\beta)} : \ell \geq 0\}$ is an orthogonal basis.

- Sobolev space $W^n = W^n_{(\alpha,\beta)}$, $n \in \mathbb{N}$, when

$$\|f\|^2_{W^n} = \|f\|^2_{W^n_{(\alpha,\beta)}} := \sum_{m=0}^{n} \left\| f^{(m)} \right\|^2_{L^2_{(\alpha+m,\beta+m)}} < \infty. \quad (15)$$

- Of course $L^2 = W^0 \supset W^1 \supset W^2 \supset \cdots$

Smoothness of a Random field

- By Kerkyacharian et al (2018), the smoothness of a Random Field is equivalent with the smoothness of the covariance (kernel).
- Lang-Schwab (2015), found the Sobolev norms for $\mathbb{S}^d$.

- Cleanthous et al (2020), express the Sobolev smoothness of the RF in terms of the angular power spectrum.

- The summability of the ps, guarantees already the $L^2_{(\alpha,\beta)}$ membership of the ck and therefore the RF.

Smoothness Theorem

Weighted ℓ^2-summability of the spectrum, is equivalent with regularity measured in terms of Sobolev spaces!

Theorem

Let $n \in \mathbb{N}$ and $\{Z(x) : x \in \mathcal{M}^d\}$ an isotropic GRF. Then $k_Z \in W^n$ if and only if

$$\|k_Z\|_{W^n}^2 \sim \sum_{\ell=0}^{\infty} \nu_\ell^2 (\ell+1)^{-2\alpha+2n-1} < \infty. \tag{16}$$

- We proved this result for the more general class of interpolation spaces, measuring non-integer smoothness.

Smooth Random fields

- On $\mathcal{M}^d = \mathbb{S}^2$, the above theorem translates as follows:

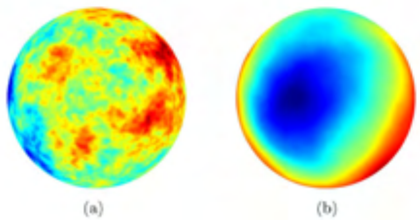
$$\|k_Z\|^2_{W^s} \sim \sum_{\ell=0}^{\infty} A_\ell^2 (\ell+1)^{2s+1} \tag{17}$$

- Simply taking $A_\ell = (\ell+1)^{-\tau}$, we have

$$k_Z \in W^s \Leftrightarrow \tau > s+1 \tag{18}$$

- Below we draw some (simulated) RFs for $\tau_1 = 3$ and $\tau_2 = 5$.

Smooth Random fields



(a) $k_Z \in W^1 \setminus W^2$ and (b) $k_Z \in W^3 \setminus W^4$.

Hölder spaces

- Let $n \in \mathbb{N}_0$. We denote by $\mathcal{C}^n = \mathcal{C}^n[-1,1]$ the set of all functions $f : [-1,1] \to \mathbb{R}$ such that their derivatives up to order n, exist and are continuous.

$$\|f\|_{\mathcal{C}^n} := \sum_{m=0}^{n} \sup_{-1 \le t \le 1} |f^{(m)}(t)| < \infty. \tag{19}$$

- Let now $N > 0$ be a non integer and let $n := [N]$ be the integer part of N. The Hölder space of order N, denoted $\mathcal{C}^N = \mathcal{C}^N[-1,1]$, is defined as the class of all functions $f \in \mathcal{C}^n$ such that

$$\|f\|_{\mathcal{C}^N} := \|f\|_{\mathcal{C}^n} + \sup_{-1 \le t \ne s \le 1} \frac{|f^{(n)}(t) - f^{(n)}(s)|}{|t-s|^{N-n}} < \infty. \tag{20}$$

Sample Hölder continuous random fields

Let $\gamma \in (0,1)$. A random field $Z : \mathcal{M}^d \times \Omega \to \mathbb{R}$ is called:

sample γ-Hölder continuous, when for every $\omega \in \Omega$, the sample function $Z(\cdot,\omega) : \mathcal{M}^d \to \mathbb{R}$ is γ-Hölder continuous, i.e. there exist a constant $c > 0$ such that

$$|Z(x,\omega) - Z(y,\omega)| \le c\rho(x,y)^\gamma, \quad \text{for every } x,y \in \mathcal{M}^d. \tag{21}$$

locally sample γ-Hölder continuous, when for every $\omega \in \Omega$, and every $z \in \mathcal{M}^d$, there exists a neighbor $V \ni z$ such that the sample function $Z(\cdot,\omega) : V \to \mathbb{R}$ is γ-Hölder continuous, i.e. there exist a constant $c = c_V > 0$ such that

$$|Z(x,\omega) - Z(y,\omega)| \le c\rho(x,y)^\gamma, \quad \text{for every } x,y \in V. \tag{22}$$

Sample Hölder continuity of the RF

Weighted ℓ^1-summability of the spectrum, implies Hölder continuity, bounds of the moments of $Z(x) - Z(y)$ and the existence of a Hölder continuous modification!

Theorem

Let $N > 0$. Let $\{Z(x) : x \in \mathcal{M}^d\}$ be an isotropic GRF, whose spectrum $(\nu_\ell)_{\ell \in \mathbb{N}_0}$ from (8) satisfies

$$\sum_{\ell=0}^{\infty} \nu_\ell (\ell + 1)^{2N} < \infty. \tag{23}$$

Then, the isotropic covariance kernel k_Z is N-Hölder continuous.

Moments of $Z(x) - Z(y)$

Theorem

Let $\{Z(x) : x \in \mathcal{M}^d\}$ *be an isotropic GRF, whose spectrum* $(\nu_\ell)_{\ell \in \mathbb{N}_0}$ *satisfies* (23) *for some* $N \in (0, 1]$*. Then, for every* $p \in \mathbb{N}$*, there exists a constant* $c = c_{N,p} > 0$ *such that for every* $x, y \in \mathcal{M}^d$*,*

$$\mathbb{E}(|Z(x) - Z(y)|^{2p}) \leq c\rho(x, y)^{2pN}. \tag{24}$$

A Kolmogorov-Chentsov type theorem

Theorem

Let $\{Z(x) : x \in \mathcal{M}^d\}$ be an isotropic GRF, whose spectrum $(\nu_\ell)_{\ell \in \mathbb{N}_0}$ satisfies (23) for some $N \in (0, 1]$. Then, there exists a continuous modification of Z which is sample Hölder continuous of order $\gamma \in (0, N)$.

Truncation of a RF

- How do we really work with an IRF?

$$Z(x) := \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\mathcal{M}^d,\ell)} \sqrt{\frac{\nu_\ell}{h(\mathcal{M}^d,\ell)}} Y_{\ell,m}(x) X_{\ell,m},$$

- Our PC?
- Truncation.

Truncation

For $r \in \mathbb{N}$, we set

$$Z^r(x) := \sum_{\ell=0}^{r} \sum_{m=1}^{h(\mathcal{M}^d,\ell)} \sqrt{\frac{\nu_\ell}{h(\mathcal{M}^d,\ell)}} Y_{\ell,m}(x) X_{\ell,m},$$

which is apparently a truncated version of the expansion (8) of the GRF, Z.

- We count the error $Z - Z^r$ in the $\mathbb{P}$-a.s and in mixed Lebesgue norms.
- The decay on the spectrum, guarantees fast approximation!

Norms for random fields

- Recall that $Z : \Omega \times \mathcal{M}^d \to \mathbb{R}$. We need to measure the integrability in the spatial domain $\mathcal{M}^d$ and the stochastic domain Ω.
- Let $p, q > 0$. We define the (quasi-)norm

$$\begin{aligned} \|Z\|_{p,q} &:= \|Z\|_{L^p(\Omega; L^q(\mathcal{M}^d))} && (25) \\ &= \left(\mathbb{E}\|Z(\cdot,\omega)\|^p_{L^q(\mathcal{M}^d)}\right)^{1/p} && (26) \\ &= \left(\mathbb{E}\left(\int_{\mathcal{M}^d} |Z(x,\omega)|^q dx\right)^{p/q}\right)^{1/p} && (27) \\ &= \left(\int_\Omega \left(\int_{\mathcal{M}^d} |Z(x,\omega)|^q dx\right)^{p/q} d\mathbb{P}(\omega)\right)^{1/p}. && (28) \end{aligned}$$

- E.g.

$$\|Z\|^2_{2,2} = \mathbb{E}\int_{\mathcal{M}^d} |Z(x,\omega)|^2 dx. \qquad (29)$$

Theorem

Let $\{Z(x) : x \in \mathcal{M}^d\}$ be an isotropic GRF, whose spectrum decays algebraically with order $1+\varepsilon$, $\varepsilon > 0$; i.e., there exist $c_* > 0$ and $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$

$$\nu_\ell \leq c_* \ell^{-1-\varepsilon}. \tag{30}$$

Then, the series of the truncated RFs $\left(Z^r\right)_r$ converges to the RF Z

1. *in $L^p(\Omega, L^2(\mathcal{M}^d))$ for every $p > 0$. Moreover there exists a constant $c = c_{p,\varepsilon} > 0$ such that*

$$\left\| Z - Z^r \right\|_{L^p(\Omega, L^2(\mathcal{M}^d))} \leq c r^{-\varepsilon/2}. \tag{31}$$

2. *$\mathbb{P}$-almost surely and for every $0 < \gamma < \varepsilon/2$, the truncated error is asymptotically bounded by*

$$\left\| Z - Z^r \right\|_{L^2(\mathcal{M}^d)} \leq r^{-\gamma}, \quad \mathbb{P}\text{-a.s.} \tag{32}$$

Multi-variate random fields on the sphere

Let $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ and $k \in \mathbb{N}$.
A k-variate random field $\mathbf{Z} : \mathbb{S}^d \times \Omega \to \mathbb{R}^k$ is called isotropic when it is of constant mean vector and

$$\mathbf{Cov}\Big(\mathbf{Z}(x), \mathbf{Z}(y)\Big) = \mathbf{C}(\rho(x, y)). \tag{33}$$

Then

$$\mathbf{C}(\theta) = \sum_{n=0}^{\infty} \mathbf{A}_n C_n^{\left(\frac{d-1}{2}\right)}(\cos\theta), \tag{34}$$

where $\mathbf{A}_n$: positive definite $k \times k$ matrices and

$$\sum_{n=0}^{\infty} \mathbf{A}_n C_n^{\left(\frac{d-1}{2}\right)}(1) \in \mathbb{R}^{k\times k}. \tag{35}$$

Karhunen-Loéve expansion: Ma (2006)

Let $\{\mathbf{V}_n\}$ sequence of independent random vectors, with $\mathbb{E}(\mathbf{V}_n) = \mathbf{0}$ and diagonal $\mathbf{Cov}(\mathbf{V}_n)$. Let $\mathbf{U}$: $(d+1)$-dimensional random vector uniformly distributed on $\mathbb{S}^d$, independent of $\{\mathbf{V}_n\}$ and $\{\mathbf{A}_n\}$ as in (35).

Then the random field

$$\mathbf{Z}(x) := \sum_{n=0}^{\infty} \mathbf{A}_n^{1/2} \mathbf{V}_n C_n^{\left(\frac{d-1}{2}\right)}(x'\mathbf{U}), \tag{36}$$

is k-variate isotropic random field of zero mean and

$$\mathbf{C}(\theta) = \sum_{n=0}^{\infty} \mathbf{A}_n C_n^{\left(\frac{d-1}{2}\right)}(\cos\theta), \tag{37}$$

Measuring matrices

- A. Alegria, P. Bisiri, G. Cleanthous, E. Porcu and P. White, Multivariate isotropic random fields on spheres: Nonparametric Bayesian modeling and L^p fast approximations. Elect. J. Stat. 2021, Vol. 15, No. 1, 2360-2392
- Let $\mathbf{A}, \mathbf{B}$ $n \times m$ matrices. The Frobenius inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle_F := \mathrm{trace}(\mathbf{A}\mathbf{B}'). \tag{38}$$

This gives the natural norm

$$\|\mathbf{A}\|_F^2 = \langle \mathbf{A}, \mathbf{A} \rangle_F = \sum_{i,j} \alpha_{ij}^2. \tag{39}$$

How do we approximate $\mathbf{Z}(x)$?

Truncation:

$$\mathbf{Z}^R(x) := \sum_{n=0}^{R} \mathbf{A}_n^{1/2} \mathbf{V}_n C_n^{\left(\frac{d-1}{2}\right)}(x'\mathbf{U}), \quad R \in \mathbb{N}. \tag{40}$$

Target: find conditions st $\mathbf{Z}^R \to \mathbf{Z}$ and measure the accuracy! Recall that in the uni-variate case we had the decay of a sequence (of numbers).

$$\begin{aligned} \|\mathbf{Z}\|_{p,2}^p = \|\mathbf{Z}\|^p_{L^p(\Omega, L^2(\mathbb{S}^d;\mathbb{R}^k))} &= \mathbb{E}\left(\|\mathbf{Z}(\cdot,\omega)\|^p_{L^2(\mathbb{S}^d;\mathbb{R}^k)}\right) \\ &= \mathbb{E}\left(\int_{\mathbb{S}^d} \|\mathbf{Z}(x)\|_F^2 dx\right)^{p/2}, \quad p \geq 1. \end{aligned} \tag{41}$$

Approximation

Theorem

Let $\mathbf{Z}(x)$ *a k-variate isotropic random field as in (36) such that*

$$trace(\mathbf{A}_n) \leq c_o n^{-d+1-\varepsilon}, \quad \text{for some } \varepsilon > 0, \tag{42}$$

then $\{\mathbf{Z}^R(x)\}_R$ *converges to* $\mathbf{Z}(x)$ *and*

$$\left\|\mathbf{Z} - \mathbf{Z}^R\right\|_p \leq C_0 R^{-\varepsilon/2}, \quad \text{for every } \ p \geq 1. \tag{43}$$

Bayesian modeling and applications

A Bayesian model for the $\mathbf{C}$.

- $\exists$ lower triangular, with nonnegative diagonal $\mathbf{B}_n$:

$$\mathbf{A}_n = \mathbf{B}_n \mathbf{B}_n'. \tag{44}$$

Propose a Bayesian model by assigning priors to $\mathbf{B}_n$'s;

The model: Let $\tilde{\mathbf{B}} := \{\tilde{\mathbf{B}}_n\}_{n \geq 0}$ independent random matrices of the same type with $\mathbf{B}_n$'s:
For every $n \geq 0$:

- iid diagonal elements
- iid off-diagonal elements
-

$$\mathbb{E}\Big((\tilde{\mathbf{B}}_n)_{11}\Big)^2 = \mathbb{E}\Big((\tilde{\mathbf{B}}_n)_{21}\Big)^2 =: d_n; \;\; \sum_{n=0}^{\infty} d_n < \infty. \tag{45}$$

The posterior

Theorem

Let $x_1, \ldots, x_n \in \mathbb{S}^d$ *and*

$$\mathbf{z} := (\mathbf{z}(x_1)', \ldots, \mathbf{z}(x_n)'), \tag{46}$$

sampled from the RF.

The posterior $\mathbb{P}^{\mathbf{z}}$ *of* $\tilde{\mathbf{B}}$*, exists, it is unique and Lipschitz continuous; small data-change, implies small changes in the posterior distribution.*

Remark: Application of the model to bivariate meteorological data (Atmospheric pressure, DSRF).

Phenomena evolving temporally

A phenomenon may present an additional time dependence; Spatiotemporal random fields.

$$Z(x, t), \; x \in \mathcal{M}, \; t \in \mathbb{T}. \tag{47}$$

- A RF $\{Z(x,t)\}$: space isotropic and time stationary when $\text{cov}(Z(x_1, t_1), Z(x_2, t_2))$ depends only on $\rho(x_1, x_2)$ and $(t_2 - t_1)$.

Spatiotemporal statistics

- G. Cleanthous, E. Porcu and P. White, Regularity and Approximation of Gaussian Random Fields Evolving Temporally over Compact Two-Point Homogeneous Spaces. TEST, (2021+).

Random fields on $\mathcal{M}^d \times \mathbb{R}$:

- Approximation
- Regularity
- Covariance modeling for the study of Ozone concentration.

- C. Doherthy.

Spatiotemporal RFs on $\mathcal{M}^d \times \mathbb{R}$.

- Let $\left\{Z(x,t):\ x \in \mathcal{M}^d,\ t \in \mathbb{R}\right\}$ space isotropic and time stationary. Then

$$\mathrm{cov}(Z(x_1,t_1), Z(x_2,t_2)) = K_{\mathrm{IS}}\Big(\cos\rho(x_1,x_2), t_2 - t_1\Big), \tag{48}$$

where the function $K_{\mathrm{IS}} : [-1,1] \times \mathbb{R} \to \mathbb{R}$ is such that

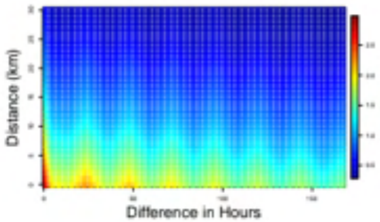
$$K_{\mathrm{IS}}(u,t) = \sum_{n=0}^{\infty} B_n(t) P_n^{(\alpha,\beta)}(u), \tag{49}$$

for a sequence B_n of stationary covariance functions;

$$\sum_{n=0}^{\infty} B_n(t) P_n^{(\alpha,\beta)}(1) \in \mathbb{R}. \tag{50}$$

Periodic phenomena

What if the spatiotemporal phenomenon present time-periodicity?
Is $\mathbb{R}$ the ideal time domain?



Periodic phenomena

Wrap the time to a circle!

$$\mathbb{S}^2 \times \mathbb{S}^1, \tag{51}$$

is the ideal domain for a spatiotemporal periodic phenomenon on the Earth.

- A. Alegria, G. Cleanthous, N, E. Porcu and P. White, Gaussian random Fields on the Hypertorus: theory and applications.

Let $d_1, d_2 \in \mathbb{N}$, we work on

$$\mathbb{T}^{d_1,d_2} := \mathbb{S}^{d_1} \times \mathbb{S}^{d_2}, \tag{52}$$

The setting includes the torus as it is isomorphic with $\mathbb{S}^1 \times \mathbb{S}^1$.

Beyond isotropy

Axial symmetry.

What is next?

- Combinations of the above.
- Other settings.
- More general manifolds.
- Isotropy?

Thank you :)

Thank you very much for your attention!