On the Excursion Area of Time Dependent Spherical Random Fields

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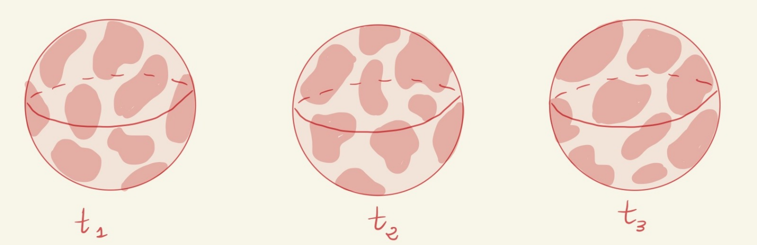
Spherical Models Applications

- **Cosmology** $\rightsquigarrow$ The Universe as seen from the Earth
  see Marinucci and Peccati (2011), *Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications*
- **Geophysics** $\rightsquigarrow$ the Earth
  see Christakos (2005), *Random Field Models in Earth Sciences*
- **Medical Imaging** $\rightsquigarrow$ the human brain
  see Taylor and Worsley (2007), *Detecting sparse signals in random fields, with an application to brain mapping*

Time Dependent Spherical Random Fields

- $Z = \left\{ Z(x,t) : x \in \mathbb{S}^2, t \in \mathbb{R} \right\}$ Gaussian random field on $(\Omega, \mathcal{F}, \mathbb{P})$
- fix t and u, we want to study $\mathcal{E}_u(t) := \left\{ x \in \mathbb{S}^2 : Z(x,t) \geq u \right\}$



Conditions on the field

- $\mathbb{E}[Z(x,t)] = 0, \quad \forall\, x \in \mathbb{S}^2, t \in \mathbb{R}$
- $\mathbb{E}\left[Z(x,t)Z(y,s)\right] = \Gamma\left(\langle x,y\rangle, t-s\right), \quad \forall\, x,y \in \mathbb{S}^2, t,s \in \mathbb{R}$
  - $\langle x,y\rangle = \cos d(x,y)$, $d(x,y)$:=angle between x and y
  - $\Gamma : [-1,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ positive semidef fct

  $\iff$ Z isotropic in space and stationary in time
- Γ continuous $\iff$ Z mean-square continuous

Multipole expansion of the field

$$Z(x,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \underbrace{a_{\ell m}(t)}_{\text{random}} \underbrace{Y_{\ell m}(x)}_{\text{deterministic}} = \sum_{\ell=0}^{\infty} Z_\ell(x,t)$$

- spherical harmonics $\{Y_{\ell m} : \ell \geq 0, m = -\ell, \dots, \ell\}$ o.b. on $L^2(\mathbb{S}^2)$
- $a_{\ell m}(t) := \int_{\mathbb{S}^2} Z(x,t) Y_{\ell m}(x)\, dx$
- $\{a_{\ell m} : \ell \geq 0, m = -\ell, \dots, \ell\}$ stationary Gaussian processes:
  - $a_{\ell m} \perp\!\!\!\perp a_{\ell' m'}$ if either $\ell \neq \ell'$ or $m \neq m'$
  - $\mathbb{E}[a_{\ell m}(t) a_{\ell m}(s)] = C_\ell(t-s)$

Representation of the covariance function

$$\Gamma\left(\langle x, y\rangle, t-s\right) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell(t-s) P_\ell(\langle x, y\rangle) = \sum_{\ell=0}^{\infty} \Gamma_\ell$$

- P_ℓ is the ℓ-th Legendre polynomial
- Γ_ℓ is the covariance function of the random field Z_ℓ
- ℓ represents the order of the so-called *multipole*

The memory of Z_ℓ

$$C_\ell(\tau) \sim C_\ell(0)\, g_{\beta_\ell}(\tau) \qquad \tau \to +\infty$$

$$g_{\beta_\ell}(\tau) = \frac{1}{(1+|\tau|)^{\beta_\ell}}$$

- long memory: $\beta_\ell \in (0,1) \rightsquigarrow$ non-integrable covariance
- short memory: $\beta_\ell \in (1,+\infty) \rightsquigarrow$ integrable covariance

The memory parameter

- β_ℓ is the memory parameter
- **assumption:** $\exists \beta_{\ell^\star} := \min\{\beta_\ell, \ell \in \mathbb{N}, \ell \geq 1\}$ (excluding β_0)
- $\beta_{\ell^\star}$ =smallest exponent=largest memory
- $\mathcal{I}^\star := \{\ell \in \mathbb{N} : \beta_\ell = \beta_{\ell^\star}\}$ ($\ell = 0$ can belong to $\mathcal{I}^\star$)
- $\mathcal{I}^\star$ are the multipoles that mostly contribute to the covariance function

Excursion area or empirical measure

- fix $u \in \mathbb{R}$
- $\mathcal{A}_u(t) := \operatorname{area}(\mathcal{E}_u(t)) = \int_{\mathbb{S}^2} \mathbf{1}_{\{Z(x,t) \geq u\}} dx$
- $\mathbb{E}[\mathcal{A}_u(t)] = 4\pi(1 - \Phi(u))$ (Φ cumulative distrib. fct. std. Gaussian)

$$\mathcal{M}_T(u) := \int_0^T (\mathcal{A}_u(t) - \mathbb{E}[\mathcal{A}_u(t)])dt$$
$$\widetilde{\mathcal{M}}_T(u) := \frac{\mathcal{M}_T(u)}{\sqrt{\operatorname{Var} \mathcal{M}_T(u)}}$$

- asymptotic behavior of $\mathcal{M}_T(u)$ as $T \to \infty$ governed by a subtle interplay between β_0, $\beta_{\ell^\star}$ and u

Main Results - Short Memory and Trivial Long Memory

- The statistic $\mathcal{M}_T(u)$ has short memory when:

$$\left.\begin{array}{l} \text{either } u \neq 0,\ \beta_0 = 1 \text{ and } 2\beta_{\ell^\star} > 1 \\[2ex] \text{or } u = 0,\ \beta_0 = 1 \text{ and } 3\beta_{\ell^\star} > 1 \end{array}\right\} \implies \begin{array}{l} \operatorname{Var} \mathcal{M}_T(u) \approx T \\[2ex] \widetilde{\mathcal{M}}_T(u) \xrightarrow{d} \text{Gaussian} \end{array}$$

- The statistic $\mathcal{M}_T(u)$ has *trivial** long memory when:

$$\left.\begin{array}{l} \text{either } u \neq 0,\ \beta_0 < 2\beta_{\ell^\star} \wedge 1 \\[2ex] \text{or } u = 0,\ \beta_0 < 3\beta_{\ell^\star} \wedge 1 \end{array}\right\} \implies \begin{array}{l} \operatorname{Var} \mathcal{M}_T(u) \approx T^{2-\beta_0} \\[2ex] \widetilde{\mathcal{M}}_T(u) \xrightarrow{d} \text{Gaussian} \end{array}$$

*the multipole $\ell = 0$ has the largest memory $\implies Z_0(x,t) = a_{00}(t)/\sqrt{4\pi}$ determines the behavior of $\mathcal{M}_T(u)$

Main Results - Long Memory

- The statistic $\mathcal{M}_T(u)$ has long memory when:
  - $u \neq 0$, and

$$\text{either } 2\beta_{\ell^\star} < \beta_0 \wedge 1 \implies \operatorname{Var} \mathcal{M}_T(u) \approx T^{2-2\beta_{\ell^\star}}$$

$$\text{or } \beta_0 = 2\beta_{\ell^\star} = 1 \implies \operatorname{Var} \mathcal{M}_T(u) \approx T \log T$$

$$\implies \widetilde{\mathcal{M}}_T(u) \xrightarrow{d} \text{non-Gaussian}$$

  - $u = 0$, and

$$\text{either } 3\beta_{\ell^\star} < \beta_0 \wedge 1 \implies \operatorname{Var} \mathcal{M}_T(u) \approx T^{2-3\beta_{\ell^\star}}$$

$$\text{or } \beta_0 = 3\beta_{\ell^\star} = 1 \implies \operatorname{Var} \mathcal{M}_T(u) \approx T \log T$$

$$\implies \widetilde{\mathcal{M}}_T(u) \xrightarrow{d} \text{non-Gaussian}$$

Sketch of the proof - Starting point

Chaotic decomposition of $\mathcal{M}_T(u)$:

- $\mathcal{M}_T(u) \in L^2(\mathbb{P})$ for any fixed T
- $\mathcal{M}_T(u) = \sum_{q=0}^{+\infty} \mathcal{M}_T(u)[q]$

$$\mathcal{M}_T(u)[q] = \frac{H_{q-1}(u)\phi(u)}{q!} \int_0^T \int_{\mathbb{S}^2} H_q\left(Z(x,t)\right) dxdt$$

- H_q is the q-th Hermite polynomial
- $\mathcal{M}_T(u)[q] := \mathrm{proj}\left[\mathcal{M}_T(u) | \mathcal{C}_q\right]$, $\mathcal{C}_q :=$ q-th Wiener chaos

An interlude on Wiener Chaos

- $q \in \mathbb{N}_0$
- $\mathbb{A} := \overline{\operatorname{lin}\{a_{\ell m}(t)\}}^{\|\cdot\|_{L^2(\mathbb{P})}}_{\ell \geq 0, |m| \leq \ell, t \in [0,T]}$
- $\mathcal{C}_q := \overline{\operatorname{lin}\{H_{p_1}(\xi_1) \cdots H_{p_k}(\xi_k)\}}^{\|\cdot\|_{L^2(\mathbb{P})}}_{p_1 + \cdots + p_k = q, \xi_1, \ldots, \xi_k \in \mathbb{A} \text{ iid} \sim \mathcal{N}(0,1)}$

$$\mathcal{C}_q \perp \mathcal{C}_{q'} \qquad q \neq q' \qquad\qquad F \in L^2(\Omega, \sigma(\mathbb{A}), \mathbb{P})$$

$$L^2(\Omega, \sigma(\mathbb{A}), \mathbb{P}) = \bigoplus_{q=0}^{+\infty} \mathcal{C}_q \quad \Longrightarrow \quad F = \sum_{q=0}^{+\infty} F[q]$$

- $\mathcal{C}_0 = \mathbb{R}$, $F[0] = \mathbb{E}[F]$, $\operatorname{Var} F = \sum_{q=0}^{+\infty} \operatorname{Var}(F[q])$

Sketch of the proof - I chaos

Study of the first chaotic component of $\mathcal{M}_T(u)$:

- $\mathcal{M}_T(u)[1]$ is Gaussian for each T:

$$\mathcal{M}_T(u)[1] = \phi(u) \int_0^T \int_{\mathbb{S}^2} Z(x,t)dxdt = \frac{e^{u^2/2}}{\sqrt{2\pi}} \int_0^T a_{00}(t)dt$$

- $\mathrm{Var}\left(\mathcal{M}_T(u)[1]\right) = \phi(u) \int_{[0,T]^2} C_0(t-s)dtds$

$$= \begin{cases} T\, \phi(u)^2 \int_{\mathbb{R}} C_0(\tau)d\tau & \beta_0 = 1 \\ T^{2-\beta_0} \dfrac{2\phi(u)^2 C_0(0)}{(1-\beta_0)(2-\beta_0)} & \beta_0 \in (0,1) \end{cases}$$

- if $\beta_0 < 2\beta_{\ell^\star} \wedge 1 \implies \widetilde{\mathcal{M}}_T(u) \sim \widetilde{\mathcal{M}}_T(u)[1] \implies$ Gaussian limit

Sketch of the proof - II chaos

Study of the second chaotic component of $\mathcal{M}_T(u)$:

- $$\mathcal{M}_T(u)[2] = \frac{H_1(u)\phi(u)}{2}\int_0^T\int_{\mathbb{S}^2} H_2\left(Z(x,t)\right)dxdt = \frac{u\phi(u)}{2}\int_0^T\int_{\mathbb{S}^2}\left(Z(x,t)^2-1\right)dxdt$$

- diagram formula:

$$\mathrm{Var}\left(\mathcal{M}_T(u)[2]\right) = \frac{u^2\phi(u)^2}{2}\int_{[0,T]^2}\int_{\mathbb{S}^2\times\mathbb{S}^2}\Gamma\left(\langle x,y\rangle, t-s\right)^2 dxdy\,dtds = \sum_{\ell=0}^{\infty}(2\ell+1)\int_{[0,T]^2} C_\ell(t-s)^2 dtds$$

Sketch of the proof - II chaos (continues)

$$u \neq 0, \quad 2\beta_{\ell^\star} < \beta_0 \wedge 1, \quad |\mathcal{I}^\star| < +\infty$$

- $\operatorname{Var}\left(\mathcal{M}_T(u)[2]\right) \sim T^{2-2\beta_{\ell^\star}} \dfrac{u^2\phi(u)^2 \sum_{\ell\in\mathcal{I}^\star}(2\ell+1)C_\ell(0)}{2(1-2\beta_{\ell^\star})(2-2\beta_{\ell^\star})}$

- $\mathcal{M}_T(u)[2] \sim \dfrac{u\phi(u)}{2} \sum_{\ell\in\mathcal{I}^\star} C_\ell(0) \sum_{m=-\ell}^{\ell} \int_0^T H_2\left(\dfrac{a_{\ell m}(t)}{\sqrt{C_\ell(0)}}\right) dt$

- $\widetilde{\mathcal{M}}_T(u) \sim \widetilde{\mathcal{M}}_T(u)[2] \xrightarrow{d}$ composite Rosenblatt distribution

Sketch of the proof - Higher chaoses

Study of higher chaoses of $\mathcal{M}_T(u)$:

- $\mathcal{M}_T(u)[q] = \frac{H_{q-1}(u)\phi(u)}{q!} \int_0^T \int_{\mathbb{S}^2} H_q\left(Z(x,t)\right) dxdt$

- diagram formula:

$$\begin{aligned}
\mathrm{Var}\left(\mathcal{M}_T(u)[q]\right) &= \frac{u^2 \phi(u)^2}{q!} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \Gamma\left(\langle x, y\rangle, t-s\right)^q \, dxdy\, dtds \\
&= \sum_{\ell_1,\dots,\ell_q=0}^{\infty} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} C_{\ell_1}(t-s) \cdots C_{\ell_q}(t-s) dtds \\
&\quad \times \frac{(2\ell_1+1)}{4\pi} P_{\ell_1}(\langle x, y\rangle) \cdots \frac{(2\ell_q+1)}{4\pi} P_{\ell_q}(\langle x, y\rangle) dxdy
\end{aligned}$$

Sketch of the proof - Higher chaoses (continues)

$$q \geq 3, \quad q\beta_{\ell^\star} < \beta_0 \wedge 1, \quad |\mathcal{I}^\star| < +\infty$$

- $\mathrm{Var}\left(\mathcal{M}_T(u)[q]\right) \sim T^{2-q\beta_{\ell^\star}} \dfrac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!(1-q\beta_{\ell^\star})(2-q\beta_{\ell^\star})}$

$$\times \sum_{\ell_1,\dots,\ell_q \in \mathcal{I}^\star} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i+1}{4\pi}} C_{\ell_i}(0) \right) \mathcal{G}^{0\dots0}_{\ell_1,\dots,\ell_q}$$

- if $u = 0$

$$\widetilde{\mathcal{M}}_T(u) \sim \widetilde{\mathcal{M}}_T(u)[3] \xrightarrow{d}$$ higher-order composite Rosenblatt distribution

Summary: Non-universality and chaoses

chaos	$u \neq 0$	$u = 0$	asympt. distrib.
I	$\beta_0 < 2\beta_{\ell^\star} \wedge 1$ $\mathrm{Var} \approx T^{2-\beta_0}$	$\beta_0 < 3\beta_{\ell^\star} \wedge 1$ $\mathrm{Var} \approx T^{2-\beta_0}$	Gaussian
II	$2\beta_{\ell^\star} < \beta_0 \wedge 1$ $\mathrm{Var} \approx T^{2-2\beta_{\ell^\star}}$	never (vanishes)	*composite* Rosenblatt
III	never	$3\beta_{\ell^\star} < \beta_0 \wedge 1$ $\mathrm{Var} \approx T^{2-3\beta_{\ell^\star}}$	*composite* Rosenblatt
all	$\beta_0 = 1,\ 2\beta_{\ell^\star} > 1$ $\mathrm{Var} \approx T$	$\beta_0 = 1,\ 3\beta_{\ell^\star} > 1$ $\mathrm{Var} \approx T$	Gaussian

Comments

- Phase transitions induced by the diagram formula:

$$\mathrm{Cov}\left(H_q(Z(x,t)), H_q(Z(y,s))\right) \approx \mathrm{Cov}\left(Z(x,t), Z(y,s)\right)^q$$

- **long memory:**
  non-summable covariance but $\exists q_0$ such that $(\text{covariance})^q$ is summable $\forall q \geq q_0 \implies$ higher chaoses do not contribute to the asymptotic behavior of $\mathcal{M}_T(u)$

- **short memory:**
  summable covariance $\implies$ $(\text{covariance})^q$ still summable
  $\implies$ all the chaoses contribute to the asymptotic behavior of $\mathcal{M}_T(u)$

Future work / Work in progress

- Studying other geometric quantities related to $\mathcal{E}_u$, with D. Marinucci (Rome) and M. Rossi (Milan);

- Testing for non-stationarity, e.g.

$$H_0: \quad Z(x,t) = \sum_{\ell m} a_{\ell m}(t) Y_{\ell m}(x) + \sum_{\ell m} \mu_{\ell m} Y_{\ell m}(x)$$

$$H_1: \quad Z(x,t) = \sum_{\ell m} a_{\ell m}(t) Y_{\ell m}(x) + \sum_{\ell m} \mu_{\ell m}(t) Y_{\ell m}(x),$$

  with A. Caponera (Lausanne) and D. Marinucci (Rome);

- Studying diffusion processes on the sphere, with G. Ascione and E. Pirozzi (Naples).

THANK YOU!